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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 5 BATON ROUGE, LA., DECEMBER, 1930 NO. 4

"TRISECTING AN ANGLE"

The tribe of "circle squarers" appears to be almost extinct. But "angle trisectors" die hard, if they die at all. Indeed at present their sophistries seem at high tide. Even the Associated Press at times give them a boost that is in no degree merited, for every purported success of an effort to divide the general angle into three equal parts by means of mere ruler and compasses is but camouflage.

Not infrequently are the headlines bold and daring. The following is a recent sample:

"TRISECTING of ARC, LONG STUMBLING BLOCK of COUNTLESS MATHEMATICIANS, IS SAID to HAVE BEEN ACCOMPLISHED."

The so-called marvelous discovery of a method of angle trisection (with the above-named instruments) is always, with never an exception, made by some one who knows nothing, or little, about mathematics. One with a reasonable knowledge of what is now classic in algebra and number theory knows that powerful analytical methods have demonstrated the impossibility of trisecting an arbitrary angle by methods restricted to the straightedge and compasses. The literature of such demonstrations is an interesting and considerable portion of the advanced mathematics courses of all standard colleges and universities in this country and in Europe.

That there have been found clever and very close approximations to the trisected values of an angle is also well-known to those properly informed. In fact, so far as this writer's experience has shown, all proposed geometric trisections have turned out to be good approximations. The really pathetic feature about it is the self-delusion entertained by the proponent when he offers to the world what he says is an absolute method of trisection because it is based upon a sound logic, when, in reality, as can be shown in every case, he is either consciously putting over some clumsily concealed fraud, or, else, is ignorant of the error he is making.

From a certain distant state, but bearing the stamp of no college or university, was recently mailed to us a pamphlet, excellent in form and typography, in which the writer, in clear Euclidean style, detailed his method of trisecting, by ruler and compasses, any angle from 0° to 180°. Struck by the system and the simplicity with which he initiated his constructions, and admiring the undoubted rigor with which he established the first seventeen links in the chain of deduction he desired to complete, we passed on to consider his eighteenth link.

It was inevitable that some link should fail to materialize. It was necessary that somewhere in his argument a false statement should be made. Here it was. We quote it verbatim.

"Now from L, draw LN parallel to VOM, meeting the circle at N".

The unbiased reader, familiar with the accepted postulate of Euclid, namely, that through ONE point not on a straight line, one line can always be drawn parallel to that straight line, must recognize at one the lack of validity of a construction which requires one to accept a postulate that one may construct a line parallel to another line by passing it through TWO POINTS not on that line. In other words the writer, plainly, had imposed THREE conditions upon a straight line, assuming them all to be INDEPENDENT conditions, when as, he should have known, only TWO independent conditions can be imposed upon it. The point N had already been constructed as the extremity of a certain radius. The point L had been ALREADY otherwise conditioned. The line VOM had already been constructed. Plainly, then, if LN was to be parallel to VOM it was not to be so by the

dictum of the writr, but only by logical necessity, if it existed.

H. L. Smith, one of our L. S. U. colleagues, to whom we referred our finding, took the trouble to parallel the trisector's (?) geometric process with a purely analytical method. He found that the logical necessity for the parallelism of the two lines (upon which the trisection was to be based) did NOT EXIST. The DIFFERENCE in the slopes of the lines he estimated to be more than 6 per cent of the *unit* of slope. *And* the estimate was an analytical one—*not* a mere measure. The lines were NOT parallel. —*S. T. S.*

JOINT ANNUAL MATHEMATICAL MEET WILL BE HELD AT NATCHITOCHES, MARCH 13, 14.

THE ATTITUDE OF THE LOUISIANA STATE DEPARTMENT OF EDUCATION TOWARD MATHEMATICS

By T. H. HARRIS
State Superintendent

If we live in the age of Science, if Number is the language of science, if Mathematics is the science of number, Mathematics logically takes a preeminent place in training for twentieth century life.

For how can a man understand science unless he comprehends its language?

Mathematics itself when wisely used condemns our ancient claim, based upon the faulty *faculty hypothesis* of mental action, to the effect that practice in the mathematics developed powers, skills, and facilities in thinking that transferred undiminished and unmodified into all fields of thought; but the mathematically determined fact that the transfer is neither in kind nor amount all that was formerly claimed for it does not give warrant for the pendulum swing that for a time did great harm to a rational understanding of the correct position of Mathematics in the public school curriculum. The initial studies of transfer of training seemed, under superficial consideration, to point to the logical conclusion that there was no transfer and that Mathematics as a non-content subject could not justify the attention given it; but further and closer study shows this inference to be no more closely related to the truth than the older *faculty hypothesis*.

The modern student of mathematical values has no warrant for concluding that we need teach in our schools only addition, subtraction, multiplication, division, and U. S. money, just because some thousands of the parents of sixth-grade children reported the use of only such arithemetical activition in their daily routine. Such a student's warrant for such a conclusion is hardly better than the warrant for concluding that a better and more adequate training in arithmetic for the aforementioned parents would have resulted both in a wider use of less simple arithmetic in their daily routine and a higher average level of self service and public service.

As we look at it in the Louisiana State Department of Education, the final settlement of the exact kinds and degrees of value in school mathematics courses is not necessary in order to warrant all that is being done at present in that field.

The development of the calculus of matrices and integral equations very obviously made possible the greater part of the remarkable work of the modern astrophysicists, and in like manner, if in smaller degree, the development in the pupils in our public-schools of a clear understanding of elementary Mathematics as the *language of science* will furnish them with the means of interpreting their environment in the age of science.

DEVELOPMENTS IN SECONDARY MATHEMATICS

By C. D. SMITH,
A. and M. College, Miss.

In the consideration of the development of secondary mathematics in America it would be interesting to sketch the history of the subject at Harvard University as a typical case. In 1693 the course at Harvard was Arithmetic and Geometry for Seniors. Algebra was an unknown science in America. In 1726, ninety years after Harvard was founded, the course was essentially the same. Professor Florian Cajori[1] has found the first record of algebra in the curriculum dated 1786. In 1787 the revised course of study placed arithmetic in the Freshman year and a pittance of algebra in the Sophomore year. In 1802 arithmetic as far as proportion was made an entrance requirement, and in 1816 a bit

of algebra was added. In 1818 we find the first full course in mathematics as follows:

First year—Algebra and Geometry
Second year—Algebra and Trigonometry
Third year—Solid Geometry and Surveying
Fourth year—Conic Sections.

In 1830 Differential Calculus was added to the Sophomore year and this course with a few minor changes was maintained until 1851. Between this and 1867 a great advance in requirements took place during which time Algebra and Geometry were placed among the requirements for admission. After 1867 Algebra thru Quadratics and Plane Geometry were made the minimum requirements for admission. This is typical of the developments in other institutions.

This record gives rise to the following interesting observation. "The subjects were stepped down until Arithmetic, a senior study in 1787, is now in the Grammar School. Algebra, too deep for a senior in 1776, is now the task of adolescents. Geometry, ministred by Euclid as strong meat for the wise men of greece, becomes milk for the babes of High School.

Another outstanding fact in regard to Algebra and Geometry during this modern era was the change in subject matter and the resulting changes in pedagogy. As mathematicians learned more and modern demands upon the sciences increased, much new material was added and many new text books were written. The result was that the early text for a senior was mild in comparison to the material found in the elementary texts of the present day. Dr. H. E. Slaught[2] states that there was quite as much poor teaching of these subjects in colleges as is now attributed to the secondary schools. It was customary for the college to lay the blame upon the preparatory school and it in turn to blame the elementary school for poor instruction. These controversies caused the Commissioner of Education to prepare a questionnaire and submit it to various institutions of every grade in an effort to locate the alleged weak points and suggest better methods. No such recommendations were made at the time. The result of these conditions probably gave rise to the great awakening and activities along these lines which came with the twentieth century. The Perry Movement in England and Dr. E. H. Moore's

presidential address before the American Mathematical Society have been accepted as the immediate stimulus of activities in the United States.

Following these events various associations of mathematics teachers were organized. The most significant of these organizations were the organizations: Teachers of Secondary Mathematics and The Mathematical Association of America. The official publication of the latter is The American Mathematical Monthly. School Science and Mathematics and The Mathematics Teacher are the official organs of large bodies of teachers. These publications have wielded a far reaching influence in the field now under discussion. They have as their purpose the distribution of a record of observations and experiences of the leaders in Modern Science and Mathematics Teaching.

Let us consider briefly a few of the outstanding results of this movement. In the "Summary of Dr. Slaught's Address" we find that a great awakening of responsibility came at this time. The first courses were confined to a more limited number of subjects and a better sequence was emphasized. The laboratory spirit in the conduct of elementary courses was originated and given great prominence. Many criticisms both from within and without the field were advanced. I will mention one of these criticisms and give in substance Dr. Slaught's reply.

One current attack was to question the right of mathematics to remain in curricula outside of technical schools basing it upon the assumption that psychologists have established the principle of "Non-transfer of Discipline". One writer said, "It has been demonstrated that the mind molded to the method of mathematics will use that method in mathematics alone". Another said, "Many practical men still talk thoughtlessly of teaching observation, honesty, precision, thoroughness etc., desiring schools to train memory, attention, judgment, reason, honor etc." Continuing he said, "Of course few thoughtful teachers of mathematics are now deceived by this out worn psychology, but the extent to which its effects still persist is discouraging to those of us who desire to have educational thinking advance beyond its present medieval stage".

Dr. Slaught's reply is substantially as follows: "We teachers of mathematics may not be psychologists but we are able to carry

over into this field enough mathematics to perceive that these arguments lead to what philosophers themselves call Reductio ad Absurdum, namely, the whole of an education is not greater than its parts since nothing from one part can be combined with another. And it leads to the further absurdity that the whole is not equal to the sum of its parts, since the parts will not blend together. Hence they would make it impossible to conceive such a thing as general culture although their followers have been wide advocates of general culture, widened vision, sound understanding etc. For if it is true that a study is useful only as applied to perpetuating itself then no study is worthwhile for the non-specialist. Professor Shorey[3] says: "I take for granted the general belief of educators, statesmen, and the man in the street, from Plato and Aristotle to Mill, Faraday, Lincoln, Taft, and France, that there is such a thing as general culture and intellectual discipline and that some studies are better mental gymnastic than others. This, like other notions of common sense, must be subject to modifications and limitations. But it is now denied altogether and these authorities are met by such names as O'Shea, Bagley, Horn, Bolton, and Degarmo. Hence tastes in authorities differ. These names have been cited not as authorities but as experts who have proved by scientific experiment that mental discipline is a myth. There are no laboratory experiments that teach us any thing about the higher mental processes which we cannot observe and infer by better methods. Still less are there any that can even approximate the solution of the complicated problem of the total value of a course of study. There is on this point no deliverance of science to oppose the vast presumption of common sense and the belief of the majority of educated and practical men."

The following conclusions might well be added to this discussion. We should be convinced by now that no subject as fundamental and far reaching as mathematics can be dropped from the curricula of non-technical schools. On the other hand we must learn how to teach it and in what sequence to teach it thru the successive stages of one's education so as to yield the maximum good to the greatest number. Due largely to the researches of Charlier, Pearson, and Rietz, a mathematical theory of statistics has been developed which is now recognized as prob-

ably the most useful of all instruments in the study of educational problems. Strange to say, altho this method was invented and developed by skillful mathematicians, they have not applied it to the solution of their own teaching problems. We should be first rather than last to use our weapons in the pedagogy of our subject. We should try to find out by this method what mathematics a student can learn, and how and when he could learn it to the best advantage. That would constitute a scientific basis for both text books and teaching. We would then know how to develop the laboratory spirit of proceeding from the known to the unknown. This is the spirit which tries to find and point out the practical, usable features of a course, that knows when to turn aside from the beaten path long enough to revel in the beauties of the wild flowers by the wayside, kindling the enthusiasm which shall spur one on thru the dull monotonous stretches of the journey.

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THE HISTORY OF THE SOLUTION OF THE CUBIC EQUATION

By LUCYE GUILBEAU
Louisiana State University

In order to trace the history of the solution of the cubic equation, it is necessary that we go back to the history of ancient peoples. Naturally, the first solutions were geometric, for ancient Egyptians and Greeks knew nothing of algebra. Both of these nations were familiar with the simple cubic equation which resulted from problems in land measuring and especially from the duplication of the cube.

The Egyptians considered the solution impossible, but the Greeks came nearer to a solution. Hippocrates of Chios, about 430 B. C., was the first to show that the duplication of a cube could be reduced to finding two mean proportionals between a given line and one twice its length, for instance, $a:x=x:y=y:2a$; since $x^2=ay$ and $y^2=2ax$, $x^4=a^2y^2$ and $x^4=2xa^3$, then $x^3=2a^3$;

but he did not find the mean proportionals. The Greeks could not solve this equation which arose, as we have said, in the problem of the duplication of a cube, and also in the trisection of an angle, by ruler and compasses, but only by mechanical curves. The foundation of the solution of cubic equations by intersecting conics was laid by the Greeks, for, it was Menaechmus, born 429 B. C., who first invented conic sections. Diophantus of Alexandria, about 300 A. D., who is preeminently the Greek writer on algebra, solved only one cubic equation, viz; the equation $x^2+2x+3=x^3+3x-3x^2-1$, which arose in the solution of a problem to find a right angled triangle such that the sum of its area and hypotenuse is a square, and its perimeter equals a cube. He did not give negative or irrational numbers as solutions, but he did give rational fractions for roots.

The Arabs improved the methods of Hippocrates of Chios and Menaechmus and at the same time developed a method originating with Diophantus and improved by the Hindoos for finding approximate roots of numerical equations by algebraic process. Al Mahani of Bagdad was the first to state the problem of Archimedes demanding the section of a sphere by a plane so that the two segments shall be a prescribed ratio in the form of a cubic equation. Abu Jafar Al Hazin was the first to solve the equation by conic sections. Abul Gud solved the equation $x^3-x^2-2x+1=0$.

The solution of cubic equations by intersecting conics was the greatest achievement of the Arabs in algebra. The foundation had been laid by the Greeks whose aim had been not to find the number corresponding to x, but simply, to determine the side x of a cube double another cube of side a. The Arabs, on the other hand, had another object in view, namely to find the roots of given numerical equations. Omar Al Hay of Chorassan, about 1079 A. D. did most to elevate to a method the solution of the algebraic equations by intersecting conics. He believed that cubics could not be solved by calculation.

The Arabs did practically no original work. They developed the work of the Greeks and the Hindoos, and by allowing themselves to be influenced more by the Greeks than by the Hindoos, they barred the road of progress for themselves.

The Hindoos did no actual work on the cubic equation, but

they developed the form and spirit of our modern algebra and arithmetic.

In Europe in 1202 Leonardo of Pisa published the Liber Abaci. This book contained all the knowledge the Arabs possessed in algebra and arithmetic and treated the subject in a free and independent way.

Leonardo was presented to Emperor Frederick II of Hohenstaufen, who was a great patron of learning. On that occasion several problems were proposed to Leonardo which he solved promptly. His methods were partly borrowed from the Arabs and partly original. One problem was the solving of $x^3+2x^2+10x=20$. As yet cubic equations had not been solved algebraically. Leonardo, changing his method of inquiry showed by clear and rigorous demonstration that the roots could not be represented by Euclidean irrational quantities; that is, constructed with ruler and compassess. He obtained close approximations to the required root.

We now pass on to Lucas Pacioli, who states in his book published in 1497 that the solution of equations such as $x^3+nx=n$ and $x^3+n=nx$ is impossible in the present state of science.

The first step in the algebraic solution of cubics was taken by Scipo Ferro (died 1526), professor of mathematics at Bologna, who solved the equation $x^3+mx=n$. Nothing more is known of his discovery than that he imparted it to his pupil Floridas in 1505. It was the practice, then, to keep discoveries secret in order to secure by that means an advantage over rivals by proposing problems beyond their reach.

A second solution is given by Nicolo of Brescia 1506 (?)—1557. When a boy he was so badly cut by a French soldier that he never regained the use of his tongue, so he was called Tartaglia, "the Stammerer". He was too poor to go to school so he learned to read and picked up, by himself, a knowledge of Latin and Greek, and mathematics. Possessing a mind of extraordinary power, he became a teacher of mathematics at one of the universities.

Tartaglia found an imperfect method in 1530 for solving $x^3+px^2=q$ but kept it secret. He spoke of his discovery in public and when Ferro's pupil, Floridas, heard of it he proclaimed his

own knowledge of the form $x^3+mx=n$. Tartaglia believing him to be a braggart challenged him to a contest, a sort of mental duel popular at that time. Then Tartaglia put forth all his zeal, industry and skill to find a rule and succeeded ten days before the contest. The difficult step was passing from the quadratic irrationals to the cubic. Placing $x=\sqrt[3]{t}-\sqrt[3]{u}$, Tartaglia perceived that irrationals disappeared from the equation making $n=t-u$. But this, together with $(m/3)^3=tu$ gives at once $t=\sqrt{(n/2)^3+(m/2)^3}+n/2$, $u=\sqrt{(n/2)^3+(m/2)^3}-n/2$. Tartaglia solved thirty problems while Floridas solved none.

In 1541 Tartaglia discovered the general solution for $x^3+px^2=+q$ by transforming it to the form $x^3+mx=+n$. This knowledge was obtained by Cardan under promise of secrecy, but he published it in his **Ars Magna**. This destroyed Tartaglia's hope of giving the world an immortal work, for the crown intended for his work had been snatched away. He challenged Cardan and his pupil Ferrari to a contest. Cardan did not appear. Tartaglia solved most of the questions in seven days, the other two did not give their results before five months and most of the solutions were incorrect.

Tartaglia then started to publish his work but he died before he reached the consideration of the cubic equation. Although he sustained his claim for priority, posterity has not conceded to him the honor of his discovery The solution is called today "Cardan's solution".

Cardan, 1501-1576, was an Italian physican and mathematician who received all advantages offered by the universities of his day. He went one step beyond Tartaglia in recognizing negative roots of an equation, calling them "fictitious".

Vieta pointed out how the construction of the roots of a cubic equation depended upon the celebrated problems of the duplication of the cube and the trisection of an angle. He reached the interesting conclusion that the former problem includes solution of all cubics in which the radical in Tartaglia's formula is real but the latter problem includes only those leading to irreducible cases.

We have seen how the cubic equation discussed by Egyptians, Greeks, and Hindoos was finally solved algebraically by the Ita-

lians of the sixteenth century. One unit of our modern algebra was discussed for practically two thousand years and from this discussion "Cardan's solution" resulted.

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THE ACTUAL AND THE ARTIFICIAL IN ANALYTIC GEOMETRY

By W. PAUL WEBBER,
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Beginners in the study of analytic geometry are not apt to be discriminating in their work. Text books contain many mechanical exercises that are intended for exercise only. That in itself is no harm. But some exercises have to do with the actual properties of curves and some do not. It would no doubt add to the effectiveness of the study of analytic geometry in schools if more attention were given to this phase.

Ordinarily we use rectangular coordinates. Is the slope of the tangent at a point fixed on the curve the same for all rectangular systems of coordinates? Obviously a rotation of axes will alter the value of the slope at a point fixed on the curve. The ordinary use of the slope does not express a characteristic of the curve any more than it does of the coordinate system in which the equation is written. On the other hand if two lines or two curves intersect at a point in one system of coordinates they will intersect at the same point relative to the curves in another system. Moreover the angle of intersection is unchanged by a change of coordinates.

When an equation expressed in one system of coordinates is transformed by a rotation of axes into another system points that were at first maximum points of the curve may no longer be maximum points in the new system. The maximum ordinate of a curve then is not independent of the coordinates. Once the attention is called to this fact it is recognized immediately.

A familiar exercise is—Show that the tangent to the para-

bola at any point is bisected at its intersection on the y-axis. This is a good enough exercise but it does not express a fundamental property of the parabola unless it is in mind that the equation is in what is called standard form. The exercise may be stated in such a way as to be true in any system of coordinates. Thus, if the principal axis of the parabola is produced through the vertex and a tangent line at any point of the curve is drawn so as to intersect the axis produced, the tangent at the vertex bisects the segment of the tangent joining the point of tangency to the point of intersection of the tangent with the principal axis. This statement is somewhat long. But would it no be worth while, at least in some cases, just to cultivate a critical attitude on the part of the student? Besides, it might be a helpful antidote for a possible overdose of "Yes and No" teaching, in earlier years.

Another familiar exercise is: Prove that the squares of the ordinates of points of the parabola are proportional to the abscissas of the same points. Stated differently this may express a characteristic of the parabola. Thus, prove that the perpendicular from any point of a parabola to its principal axis is proportional to the distance from the vertex to the foot of the perpendicular. In this form the exercise has a meaning for any system of coordinates.

The relation of pole and polar of a conic is a characteristic of the curve and is therefore independent of the system of coordinates in which such relation is expressed.

It is not to be supposed that we shall eliminate all exercises that do not have to do with actual properties of curves. It is our purpose only to mention the desirability of calling attention to the matter in teaching so that the student may not be left with a false impression or be left entirely unconscious of the distinction. In fact many of the artificial exercises are useful in solving real problems. We can hardly dispense with the derivative as the slope but we should be conscious that the slope at the same point of a curve may have different values for different systems of coordinates.

Another set of exercises are based on the idea of the lengths of the tangent, sub-tangent, normal and sub-normal. Theses quantities are interesting, and useful for some purposes,

but they are not strictly properties of curves. They are dependent on the coordinate system in which they are expressed.

One might go on enumerating such cases indefinitely, but enough has been indicated to show that the distinction between actual characteristics of curves and characteristics which grow out of the coordinate systems used is a subject of some interest.

THE CIRCULAR AND HYPERBOLIC FUNCTIONS

By H. L. SMITH,
Louisiana State University

It is well known that the properties of the circular functions and the properties of the hyperbolic function are closely analogous throughout. But in spite of this analogy, the methods of treatment of these functions in elementary mathematics are not at all alike, the treatment of the circular functions being geometric in character while the treatment of the hyperbolic functions is purely analytic, being based on the exponential function.

In the present note the properties of both classes of functions are derived by a common geometric method. This makes it possible to introduce the exponential function and develop its properties in a very simple fashion.

§1. A certain geometric method of defining pairs of functions by means of the notion of area. In a plane set up a rectangular coordinate system. Let C be a connected, continuous curve that satisfies the following conditions: (1) every half-line with O, the origin, as end-point intersects the curve C in at most one point; (2) the positive x-axis cuts C once, say it U; (3) if C cuts the negative x-axis, then C is closed.

Two points P, Q of C determine a single arc on C, denoted by arc P Q, unless C is closed; in this case we denote by arc P Q that one of the arcs determined by P and Q which does not cut the negative x-axis. By sector OPQ will be meant the set of points bounded by arc PQ and line segments OP, OQ.

We now set up on C a scale (coordinate system) as follows. Let P be any point on C not on the x-axis, and let

$t = +2$(area sector OUP),

the sign being + or minus according as the ordinate of P is positive or negative. To the point P assign the scale-number (coordinate) t. To the point U assign the scale number 0. If C cuts the negative x-axis, it cuts it in a single point, say V; to V we assign the scale-number

2(area sector O U W+area sector O W V),

where W is the point in which C cuts the positive y-axis.

In this fashion one, and only one, scale-number t is assigned to every point P on C, and the correspondence between P and t is continuous, except at V in the case that C is closed. In this case we remove the discontinuity by assigning to every point P in addition to the scale-number t an infinity of others given by the formula

$$t+2nA$$

where n is any positive or negative integer and A is the area of C.

Now let (x, y) be the coordinates of any point P of C. Then x and y are single-valued functions of the scale-number t of P. It is also clear that if C is closed x and y are periodie functions, with period 2A, of t.

§2. **An Approximate formula for $\triangle t$, where t is as in §1.** Let P and Q be two points of C having coordinates $(x, y.)$, $(x+\triangle x, y+\triangle y)$, respectively, and having scale-numbers t and $t+\triangle t$ respectively. We suppose that both P and Q are distinct from U and V and that arc P Q does not contain U. Then

(1) $\triangle t = sgn\ (P, Q)\ (sgn\ y)\ 2$ (area sector O P Q),

where $sgn\ (P, Q) = +1$ or -1 according as Q is not or is on arc U P and $sgn\ y = +1$ or -1 according as y is postive or negative.

We now determine the value of $sgn\ (P\ Q)$. Obviously sgn (P, Q) is +1 or —1 acording as Q and U are on opposite or on the same side of O P. Set

$$F\ (XY) = xY - yX$$

Then if $U = (a, 0)$, Q and U are on opposite or on the same side or 0 P according as $F(x+\triangle x, y+\triangle y)\ F(a, 0)$ is negative or positive. Hence

$$sgn\ (P, Q) = -sgn\ F(x+\triangle x, y+\triangle y)\ sgn\ F\ (a, 0)$$

But

$$sgn\ F(x+\triangle x, y+\triangle y) = sgn\ (x\triangle y - y\triangle x)$$

$$sgn\ F(a,0) = sgn\ (-ay) = -sgn\ y.$$

Hence

(2) $sgn\ (P, Q) = sgn\ (x\triangle y - y\triangle x)\ sgn\ y.$

From (1) and (2),

(3) $\triangle t = sgn(x\triangle y - y\triangle x)$. 2 (area sector O P Q)

But if Q is close to P the area of triangle O P Q is a close approximation to that of sector O P Q. Hence

(4) area sector O P Q$=\frac{1}{2}(x\triangle y - y\triangle x)$, nearly

From (3), (4)

$\triangle t = sgn(x\triangle y - y\triangle x).\ (x\triangle y - y\triangle x)$, nearly,

or

(5) $\triangle t = x\triangle y - y\triangle x$, nearly,

which is the desired approximate formula.

The formula (5) was obtained on the supposition that P is not on the x-axis; the case in which this is not true is easily treated.

§3. **On the degree of approximation of the formula of §2.** Let r=OP, T=circular measure of angle P O Q and let $\triangle r$ be the smallest value of h for which are P Q has no points outside the circle with centre at O and radius $r+h$ and no points inside the circle with centre at O and radius $r-h$. Then

(1) $\frac{1}{2}T(r-\triangle r)^2 \leqq$ area sector O P Q $\leqq \frac{1}{2}T(r+\triangle r)^2$

$\frac{1}{2}T(r-\triangle r)^2 \leqq$ area triangle O P Q $\leqq \frac{1}{2}T(r+\triangle r)^2$,

since both the sector O P Q and triangle O P Q contain a circular sector with angle T and radius $r-\triangle r$ and both are contained in a circular sector of angle T and radius $r+\triangle r$. Hence, by (1)

$$\left(\frac{r-\triangle r}{r+\triangle r}\right)^2 \leqq \frac{\text{area triangle O P Q}}{\text{area sector O P Q}} \leqq \left(\frac{r+\triangle r}{r-\triangle r}\right)^2$$

or

(2) $$\left(\frac{r-\triangle r}{r+\triangle r}\right)^2 \leqq \frac{x\triangle y - y\triangle x}{\triangle t} \leqq \left(\frac{r+\triangle r}{r-\triangle r}\right)^2$$

Now let Q approach P along C. Then $\triangle r$ approaches zero and it follows from (2) that $(x\triangle y - y\triangle x)/\triangle t$ approaches 1. Hence we may write

(3) $x\triangle y - y\triangle x = (1+e)\triangle t$,

where e approaches zero with $\triangle t$.

§4. **Differentiation of the functions x and y of §1.** We are now in a position to differentiate the functions x and y of §1. Suppose the equation of C can be written

(1) $f(x) = g(y)$

and that x and g are both differentiable functions. At Q we have

(2) $f(x+\triangle x)=g(y+\triangle y)$

From (1), (2) we have

(3) $(\triangle f/\triangle x)(\triangle x/\triangle t)-(\triangle g/\triangle y)(\triangle y/\triangle t)=0$

where

$\triangle f=(x+\triangle x)-f(\mathrm{x}), \quad \triangle g=g(y+\triangle y-g(y).$

From (1), (2) we have

(4) $y(\triangle x/\triangle t)-x(\triangle y/\triangle t)=-(1+e)$

On solving (3) and (4) simultaneously, we get

$\triangle x/\triangle t=(1+e)(\triangle y/\triangle t)/[x(\triangle x/\triangle t)-y(\triangle y/\triangle t)],$

$\triangle y/\triangle t=(1+e)(\triangle x/\triangle t)/[x(\triangle x/\triangle t)-y(\triangle y/\triangle t)].$

On letting $\triangle t$ approach zero, we get

(5) $dx/dt=g^1(y)/[xf^1(x)-yg^1(y)]$

$dy/dt=f^1(y)/[xf^1(x)-yg^1(y)],$

provided of course the common denominator is not zero. We have thus obtained the desired formulas.

§5. The function sin t and cos t. Let the curve C of §1 be taken to be the circle,

(1) $x^2=1-y^2.$

and write

(2) $x=\cos t, \; y=\sin t,$

which are to be regarded as defining the cos and sin functions. Then we have at once by (1)

(3) $\cos^2 t+\sin^2 t=1.$

Also we have easily from the definition of t and elementary properties of the circle,

(4) $\cos 0=1, \; \sin 0=0$

$\cos(-t)=\cos t, \; \sin(-t)=-\sin t.$

Also by (1) and §4, eq. (5), we have

(5) $(d/dt)\cos t=-\sin t$

$(d/dt)\sin t=\cos t$

§6. The addition theorems for the functions sin t, cos t. Consider the function

(1) $\mathrm{F}(t)=\cos(a-b+t)\cos t+\sin(a-b+t)\sin t.$

By (5), §5 we get

$\mathrm{F}^1(t)=0$

Hence $\mathrm{F}(t)$ is contant so that

(2) $\mathrm{F}(0)=\mathrm{F}(b)$

But by (4), §5 the equation (2) reduces to

(3) $\cos(a-b)=\cos a\cos b+\sin a\sin b$

On replacing b by $-b$ in (3) and using (4), §5, we get

(4) $\cos(a+b) = \cos a \cos b - \sin a \sin b$,

which is the cosine addition theorem.

Similarly on starting with

$G(t) = \sin(a-b+t)\cos t - \cos(a-b+t)\sin t$,

we obtain

(5) $\sin(a-b) = \sin a \cos b - \cos a \sin b$

and hence

(6) $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

§7. **The functions sinh t and cosh t.** Now let the Curve C of §1 be the branch of the hyperabola

(1) $x^2 = 1 + y^2$

for which

(2) $x > 0$,

and let sinh t, cosh t be definied by

(3) $x = \cosh t$, $y - \sinh t$.

Then by (1)

(4) $\cosh^2 t - \sinh^2 t = 1$.

It also follows from (1), (2) that

(5) $\cosh t + \sinh t > 0$

Moreover

(6) $\cosh 0 = 1$, $\sinh 0 = 0$

$\cosh(-t) = \cosh t$, $\sinh(-t) = -\sinh t$

Finally by (5), §5 and (1)

(7) $(d/dt)\cosh t = \sinh t$

$(d/dt)\sinh t = \cosh t$

§8. **The addition theorems for the functions sinh t, cosh t.** By aid of the function

$F(t) = \cosh(a-b+t)\cosh t - \sinh(a-b+t)\sinh t$

we find as in §6, the formula

(1) $\cosh(a-b) = \cosh a \cosh b - \sinh a \sinh b$,

from which

(2) $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$,

which is the addition theorem for the hyperbolic cosine. By aid of the function

$G(t) = \sinh(a-b+t)\cosh t - \cosh \mathrm{A}a-b+t)\sinh t$,

we find

(3) $\sinh(a-b) = \sinh a \cosh b - \cosh a \sinh b$,

from which

(4) $\sinh(a+b)=\sinh a \cosh b+\cosh a \sinh b$,
which is the addition theorem for the hypherbolic sine.

§9. **The exponential function.** By aid of (2), (4), §8, the formula

(1) $(\cosh t+\sinh t)^r=\cosh rt+\sinh rt$

is easily established, by the method of mathematical induction, for postive integral values of r. Then by aid of (5), §7 the truth of (1) for all positive rational values may be seen. Also by aid of (4), (6), §7, its truth for all rational values of r may be established It then follows that (1) is true for all real values of r since both its members are continues function of r.

In (1) put $t=1$, $r=t$, getting

(2) $(\cosh 1+\sinh 1)^t=\cosh t+\sinh t$

Now set

(3) $e=\cosh 1+\sinh 1$

Then (2) becomes

(4) $e^t=\cosh t+\sinh t$

which defines the exponential function.
By (4) and (7), §7 we obtain

(5) $(d/dt)e^t=e^t$

SOME ASPECTS OF PRIMES AND INTEGERS

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I. The Number of Primes in Certain Intervals.

In our paper "Relative Primality," published in the November issue of the Mathematics News Letter, we established and used the theorem that the number of integers relatively prime to and less than $n=p^a q^b \ldots v^i$ is expressed by (1) $P=(p-1)(q-1)\ldots(v-1)\,p^{a-1}q^{a-1}\ldots v^{i-1}$,

where $p, q, \ldots v$, are distinct primes and a, b, i are positive integers. We showed that if n is not too large the formula may be used with but little labor to compute the number of primes in certain intervals. For example, we showed that there are 82 absolute primes less than 420.

We proceed to use the formula to determine the number of

absolute primes between 420 and $840=2^3.3.5.7$

Applying (1), with $n=840$, we have

$$P=4.2.4.6=192,$$

which is the number of integers less than 840 that do not contain 2, 3, 5, 7 as factors. Since, as shown in our previous article, the value of P for $n=840$ is twice the value of P for $n=420$, it follows that the number of integers between 420 and 840 not having the factors 2, 3, 5, 7, s 96.

By estimating the number, C, of composite integers free of factors 2, 3, 5, 7, in this interval, we shall have in the same interval

96—C = the number of absolute primes,

We proceed to estimate C.

No integer of the C composites has a prime factor less than 11, nor one greater than 840/11 or 73. Furthermore, since the first prime greater than 420/11 is 41, it is plain that the number of absolute primes, 96—C in the interval from 420 to 840, is determined when we determine the number of products greater than 420 and less than 840 which may be made from the two sets, namely, (2), 11, 13, 17, 19, 23, 29, 31, 37

and, (3), 41, 43, 47, 53, 59, 61, 67, 71, 73.

(a) Since $19^2<420$, and $29^2>840$

the only perfect square of the C composites is

$$23^2$$

(b) There are 9 composites of C having 11 as a factor.

(c) Since, approximately $840/13=61$, and $420/13=37$, there are 7 composites of C having 13 as a factor.

(d) By similar considerations there are 6 numbers of C having 17 as a factor.

(e) There are 6 numbers of C having 19 as a factor.

(f) There are 2 numbers having 23 as a factor, namely, 23·29, 23·31.

(g) No composite of C has more than two prime factors.

Summing: $C=1+9+7+6+6+2$

$=31$

$\therefore 96—C=96—31$

or the number of absolute primes in the interval from 420 to 840 is 65, or, combining with former result there are 65+82=-

147 primes less than 840.

Again let $n=2^4{\cdot}3{\cdot}5{\cdot}7=1680$

Then $\qquad P=8{\cdot}2{\cdot}4{\cdot}6=384$

and there are 192 integers greater than 840 and less than 1680, not having 2, 3, 5, 7 as a factor.

Using a condensed formulation of our method but proceeding precisely as before, we have the following sets of primes:

(1) 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73

(2) 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151

We classify as follows all those powers and products of (1) and (2) which are greater than 840 and less than 1680.

(a) One cube, 11^3

(b) 3 squares, 29^2, 31^2, 37^2

(c) Having 11 and one larger distinct prime factor, 15

(d) Having 13 and one larger distinct prime factor, 14

(e) Having 17 and one larger distinct prime factor, 10

(f) Having 19 and one larger distinct prime factor, 9

(g) Having 23 and one larger distinct factor, 10

(h) Having 29 and one larger distinct factor, 6

(i) Having 31 and one larger distinct factor, 5

(j) Having 37 and one larger distinct factor, 2

Thus $C=75$ from which $192-75=117$, which is the number of absolute primes greater than 840 and less than 1680. Thus there are $147+75$, or, 222 absolute primes less than 1680.

II. The Forms $pq+p+q=S$, p, q, S *Prime*,

While it is well known that no general formula for a prime number has ever been discovered, the following tables exhibit remarkable prime properties of the form $pq+p+q$ in which it is assumed that p, q are known primes. The tables are offered as showing that for a remarkably large number of primes p, q, at least one of the four forms has an odd prime value, S, namely, (a), $pq+p+q$, (b) $pq-p-q$, (c) $pq+p-q$, (d) $pq-p+q$.

Table 1 (S *arbitrary*)

$pq+p-q=S$	$pq+p+q=S$
$pq-p+q=S$	$pq-p-q=S$
$1{\cdot}2-2+1=1$	$2{\cdot}17+2+17=53$
$2{\cdot}1+2-1=3$	$2{\cdot}19+2+19=59$
$2{\cdot}3-3+2=5$	$2{\cdot}59+2-59=61$

$1\cdot3+3+1=7$	$2\cdot23+2+23=71$
$2\cdot3+3+2=11$	$2\cdot71+2-71=73$
$3\cdot5-5+3=13$	$3\cdot19+3+19=79$
$3\cdot7-7+3=17$	$5\cdot13+5+13=83$
$2\cdot7+7-2=19$	$3\cdot23-3+23=89$
$3\cdot5+5+3=23$	$3\cdot53-3-53=103$
$3\cdot13+3-13=29$	$5\cdot17-5+17=97$
$3\cdot7+3+7=31$	$2\cdot103-2-103=101$
$2\cdot13-2+13=37$	$7\cdot19-7-19=107$
$2\cdot13+2+13=41$	$5\cdot19-5+19=109$
$2\cdot41+2-41=43$	$3\cdot29-3+29=113$
$3\cdot11+3+11=47$	etc. etc. etc. (?)

Several comments may be made on the above tabulations

(1) No primes p and q exist such that $pq+p+q=2$. The proof that this is true falls under two heads, namely,

Case (a), when p is 2, q an odd prime,

Case (b), when p and q are both odd primes.

Case (a), when $p=2$ and q is odd $pq+p$ is even, making $pq+p+q=$an odd number.

Case (b), As p and q are both odd pq is odd, so that $pq+p=$ an even number. Hence $pq+p+q=$an odd number. The proof is valid for the four forms.

Thus $(pq+p+q)$ is not 2, for any primes $p, q, -p, -q$.

If p and q were not restricted to prime values there would be indefinitely many solutions of $pq+p+q=S$, for each value assigned to S. This suggests the inquiry: Does this restriction to prime values result in a corresponding restriction of the number of solutions?

We leave the question to be considered by any interested reader. However, trial will show that many of the above values of S may be obtained by other assignments to p and q than those given.

Table 2

(p, q, arbitrary primes, one or both odd)

$5\cdot23+5-23=97$, prime

$7\cdot23+7+23=191$, prime

$7\cdot29+7+29=239$, prime

$7\cdot31-7+31=241$, prime

$7\cdot37+7-37=229$, prime

$11\cdot 29+11+29=359$, prime
$2\cdot 37+2+37=113$, prime
$2\cdot 53-2+53=157$, prime
$13\cdot 37+13-37=457$, prime
$17\cdot 41+17-41=673$, prime
$17\cdot 47+17+47=863$, prime
$23\cdot 53-23+53=1249$, prime
$29\cdot 59-29+59=1741$, prime
$61\cdot 71+61+71=4463$, prime
$73\cdot 29-29+73=2161$, prime
$61\cdot 73+61-73=4441$, prime
$41\cdot 97+41+97=5\cdot 823$, not prime
$41\cdot 97-41-97=11\cdot 359$, not prime
$41\cdot 97-41+97=19.307$, not prime
$41\cdot 97+41-97=3\cdot 1307$, not prime

NOTE ON A NEW BOOK

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The successful teacher must above all be a master of the subject which he teaches, but he, also must know how so to teach his subject that his students may master the subject in the limited time allotted—in brief, he must be familiar with modern tendencies and general aims and endowed with a keen insight into innate characteristics of human nature, in order that he may arouse and keep the interest of his students. Skilled teachers in the past few years have been making definite studies whereby the best results may be attained and are giving valuable aid to the mass of the teaching profession thru their writings. Of significant importance is the book reviewed below.

First year Algebra-Manual for Teachers, by Stone-Mallory. Publishers: Benj. H. Sanborn & Co., Chicago, 1930.

Chapter I treats of General Objectives, The Learning Process, The Verbal Problem, and, under five important heads, Teaching Technique.

The chief objective and the most valid one is, that a course in mathematics should give the pupil "proper habits of thinking".

The author quotes Morrison, "The Practice of Teaching in the Secondary School." The conditioning factors under which thinking takes place are: (1) Something to think about; (2) A method of thinking; (3) Inherent capacity to think at all; and (4) A motive for thinking.

The author points out that Mathematics is primarily a **method of thinking** and the mastery of each unit of subject-matter puts the student in possession of an additional tool for use in reflective processes. The main objectives set forth are: developing proper habits of thinking, fixing desirable mental attitudes,—attitudes of independence, resourcefulness, inquiry after truth, insistence on accuracy, and quantitative thinking.

The learning process requires a definite teaching cycle as the author so forcefully illustrates. A psychological procedure is outlined; first, the introductory presentation of the formula; next, the practice exercises in making and using the formula; then, a thorough review coordinating all the work on the formula. At increasing intervals of time and in gradually decreasing amounts, the work is reviewed in settings gradually becoming more complex. At the end of a regular interval this material, together with other units of work studied, receives a thorough review.

How to meet the pupils' needs and how to conduct a class of differentiated groups with maximum, medium, and minimum assignments are also discussed. A definite procedure is outlined for teaching the verbal problem, for the interpretation of problems is one of the big pitfalls in the teaching of algebra. Several points of teaching technique are suggested and would be of much benefit to the beginning teacher of algebra.

The testing program outlined in Chapter II is of significant value, for it analyzes errors made by pupils, their frequency, their causes and how to forestall them. It is undoubtedly a very graphic presentation.

Suggestions for best presentation of each unit of work and why it is best with helpful suggestions, devices, etc. are most graphically set forth.

This would indeed be a most helpful manual in the hands of any teacher of algebra, whether a beginning teacher who is seeking help, or the self-satisfied, seasoned professor, for he, too, can find valuable helps for bettering his good teaching.